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On asymptotically flat space–times

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Abstract. Relations between harmonic and radiating coordinates are investigated for the case of asymptotically flat space–times, and an approximation to the 'Bondi news function' in terms of the energy–momentum tensor is found.

1. Introduction

In investigating the behaviour of space–times at large distances from bounded sources, several authors have used null radiating coordinates (Newman and Unti 1963, Bondi *et al* 1962, Papapetrou 1969). The interpretation of the behaviour of space–time in terms of the behaviour of its sources is difficult to achieve in radiating coordinates, and so we would like to have transformations from radiating coordinates to other coordinates in which the interpretation is easier.

In § 2 of this paper a derivation is given of the relations between the null radiating coordinates of Newman and Unti (1963) and harmonic coordinates. Relations between radiating coordinates and harmonic coordinates have previously been investigated by Isaacson and Winicour (1968) and by Madore (1970, and references therein), whose methods we follow fairly closely.

In § 3, using the results of § 2, we obtain a formula for the 'Bondi news function' in terms of quantities dictating the behaviour of the metric in harmonic coordinates, this formula having been previously obtained by Madore (1970) using a different technique. Finally, using this formula we are able to express the news function in terms of the energy–momentum tensor.

2. Transformations between harmonic and radiating coordinates

The solutions of Newman and Unti (1963) in null radiating coordinates (u, r, θ, ϕ) are given by:

$$
\begin{aligned}
g^{ur} &= 1, \qquad g^{uu} = g^{\theta u} = g^{\phi u} = 0, \\
g^{rr} &= -1-(\psi_2^0+\bar{\psi}_2^0)r^{-1}+\mathrm{O}(r^{-2}), \\
g^{r\theta} &= \mathrm{Re}\,(\eth\sigma^0)r^{-2}+\mathrm{O}(r^{-3}), \\
g^{r\phi} &= \mathrm{Im}\,(\bar{\eth}\sigma^0)r^{-2}+\mathrm{O}(r^{-3}), \\
g^{\theta\theta} &= -r^{-2}+2\,\mathrm{Re}\,\sigma^0 r^{-3}+\mathrm{O}(r^{-4}),
\end{aligned}
$$

$$g^{\phi\phi} = \frac{-r^{-2}}{\sin^2\theta} - \frac{2\,\mathrm{Re}\,\sigma^0}{\sin^2\theta} r^{-3} + \mathrm{O}(r^{-4}),$$

$$g^{\theta\phi} = \frac{\mathrm{Im}\,\sigma^0}{\sin\theta} r^{-3} + \mathrm{O}(r^{-4}),$$

where

$$\frac{\partial \psi_2^0}{\partial u} + \frac{1}{2}\eth\bar{\eth}\left(\frac{\partial \bar{\sigma}^0}{\partial u}\right) + \sigma^0 \frac{\partial^2 \bar{\sigma}^0}{\partial u^2} = 0,$$

$$\psi_2^0 - \bar{\psi}_2^0 = \tfrac{1}{2}\bar{\eth}\bar{\eth}\sigma^0 - \tfrac{1}{2}\eth\eth\bar{\sigma}^0 + \bar{\sigma}^0 \frac{\partial \sigma^0}{\partial u} - \sigma^0 \frac{\partial \sigma^0}{\partial u},$$

ψ_2^0, σ^0 are independent of r and $\sigma^0(u, \theta, \phi)$ is an arbitrary bounded function.

The wave equation

$$\Phi^{;\nu}{}_{;\nu} = 0,$$

with the above metric and assuming that

$$\Phi, \frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi}, \frac{\partial^2 \Phi}{\partial \theta \partial \phi}, \frac{\partial^2 \Phi}{\partial \theta^2}, \frac{\partial^2 \Phi}{\partial \phi^2} = \mathrm{O}(r),$$

$$\frac{\partial \Phi}{\partial r} = \mathrm{O}(1) \qquad \text{and} \qquad \frac{\partial^2 \Phi}{\partial r^2} = \mathrm{O}(r^{-1}),$$

becomes, using the results of Newman and Unti (1963),

$$\begin{aligned}
-[1+(\psi_2^0+\bar{\psi}_2^0)r^{-1}]\frac{\partial^2 \Phi}{\partial r^2} + \frac{2\partial^2 \Phi}{\partial u \partial r} &+ r^{-1}\left(-[1+(\psi_2^0+\bar{\psi}_2^0)r^{-1}]\frac{\partial}{\partial r} + \frac{2\partial}{\partial u}\right)\Phi \\
&+ \left(-r^{-1} + (\psi_2^0-\bar{\psi}_2^0)r^{-1} - 2\bar{\sigma}^0 \frac{\partial \sigma^0}{\partial u} r^{-2}\right)\frac{\partial \Phi}{\partial r} \\
&- r^{-2}\bar{\eth}\eth\Phi - r^{-2}\bar{\eth}\left(\bar{\eth}\sigma^0 \frac{\partial \Phi}{\partial r}\right) + r^{-3}\sigma^0\bar{\eth}\bar{\eth}\Phi \\
&+ r^{-3}\bar{\eth}\sigma^0\bar{\eth}\Phi + r^{-3}\bar{\sigma}^0\eth\eth\Phi + \mathrm{O}(r^{-3}) = 0
\end{aligned} \tag{1}$$

(for definitions and properties of the $\eth$, $\bar{\eth}$ operators see Newman and Penrose 1966).

Taking $\bar{x}^\alpha$ to be harmonic coordinates, they satisfy

$$(\bar{x}^\alpha)^{;\nu}{}_{;\nu} = 0, \tag{2}$$

and the metric components in harmonic coordinates $\bar{g}^{\alpha\beta}$ are given by

$$\bar{g}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} g^{\mu\nu}.$$

On taking the quantities $\bar{x}^\alpha$ to satisfy the same order of magnitude conditions as Φ, we seek an asymptotic solution to (2) of the form

$$\bar{x}^\alpha \sim A^\alpha + B^\alpha u + C^\alpha \ln r + D^\alpha r + E^\alpha r^{-1} \ln r + F^\alpha r^{-1} \tag{3}$$

where : (*a*) A^α is constant; (*b*) $B^\alpha, C^\alpha, D^\alpha$ are independent of r and u; and (*c*) E^α and F^α are independent of r. On substitution of (3) in (2) we obtain: (*a*) B^α is constant; (*b*) $\bar{\eth}\eth B = -2B+2A$; and (*c*)

$$2\frac{\partial E^\alpha}{\partial u}-C^\alpha-2D^\alpha\left(\bar{\Psi}_2^0+\bar{\sigma}^0\frac{\partial \sigma^0}{\partial u}+\tfrac{1}{2}\bar{\eth}\bar{\eth}\sigma^0\right)-\bar{\eth}\eth A^\alpha = 0.$$

We assume that when

$$Q = \bar{\Psi}_2^0+\bar{\sigma}^0\frac{\partial\sigma^0}{\partial u}+\tfrac{1}{2}\bar{\eth}\bar{\eth}\sigma^0$$

there exist an m independent of u such that

$$\frac{1}{4\pi}\iint Q\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \to -m \qquad \text{as } u\to\infty \text{ (or } -\infty),$$

and where integration is over the unit sphere. Defining ΔQ by

$$\Delta Q = Q+m$$

we may take for transformations (3)

$$\bar{x}^0 \sim u+r+2m\ln r+\left(\int_\infty^u \Delta Q\,\mathrm{d}u\right)r^{-1}\ln r+E^0r^{-1}$$

$$\bar{x}^a \sim Rn^a+E^ar^{-1}$$

where

$$n^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

and

$$R = r-m+\left(\int_\infty^u \Delta Q\,\mathrm{d}u\right)r^{-1}\ln r.$$

The metric in harmonic coordinates now takes the form

$$\bar{g}^{\mu\nu} = n^{\mu\nu}+h^{\mu\nu}r^{-1}+2\Delta Qk^\mu k^\nu r^{-1}\ln r$$

and the metric density is given by

$$(-\bar{g})^{1/2}\bar{g}^{\mu\nu} = n^{\mu\nu}+b^{\mu\nu}r^{-1}+2\Delta Qk^\mu k^\nu r^{-1}\ln r \tag{4}$$

where $k^\mu = (1, n^a)$, $n^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $b^{\mu\nu}, h^{\mu\nu}$ are independent of r. (It should be noted that Greek indices range from 0 to 3 and Latin indices from 1 to 3.)

3. The news function

In obtaining their solutions Newman and Unti (1963) used the complex tetrad $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ where

$$l^\mu n_\mu = -\bar{m}_\mu m^\mu = 1$$

(all other products zero), and

$$g^{\alpha\beta} = 2l^{(\alpha}n^{\beta)}-2\bar{m}^{(\alpha}m^{\beta)}.$$

Newman and Unti found that

$$m^{\mu} = \left(0, \mathrm{O}(r^{-1}), \frac{1}{\sqrt{2}}r^{-1} + \mathrm{O}(r^{-2}), \frac{\mathrm{i}r^{-1}}{\sin\theta} + \mathrm{O}(r^{-2})\right),$$
$$l^{\mu} = (0, 1, 0, 0),$$
$$n^{\mu} = (1, -\tfrac{1}{2} + \mathrm{O}(r^{-1}), \mathrm{O}(r^{-3}), \mathrm{O}(r^{-3}))$$

and that the scalar

$$\psi_4 = -C_{\alpha\beta\gamma\delta}n^{\alpha}\overline{m}^{\beta}n^{\gamma}\overline{m}^{\delta} = -\frac{\partial^2\bar{\sigma}^0}{\partial u^2}r^{-1} + \mathrm{O}(r^{-2}) \tag{5}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl conformal tensor expressed in null coordinates. On transforming ψ_4 to harmonic coordinates, we obtain

$$\psi_4 = -\mathcal{C}_{\alpha\beta\gamma\delta}b^{\alpha}\bar{c}^{\beta}b^{\gamma}\bar{c}^{\delta} \tag{6}$$

where

$$b^{\mu} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\nu}}n^{\nu} \sim \tfrac{1}{2}(1, -n^{a}),$$

$$c^{\mu} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\nu}}m^{\nu} \sim \frac{1}{\sqrt{2}}(0, \eth n^{a})$$

and $\mathcal{C}_{\alpha\beta\gamma\delta}$ is the Weyl tensor expressed in harmonic coordinates.

The asymptotic behaviour of the Weyl tensor in harmonic coordinates, on using (4), is given by

$$\mathcal{C}_{\alpha\beta\gamma\delta} \sim -\tfrac{1}{2}r^{-1}k_{[\alpha}\partial^2 b_{\beta][\gamma}k_{\delta]},$$

where $b_{\alpha\beta} = b^{\gamma\delta}n_{\gamma\alpha}n_{\delta\beta}$, and as a consequence we will have from (6) that

$$\psi_4 \sim -\frac{1}{2}r^{-1}\bar{c}^{\alpha}\bar{c}^{\beta}\frac{\partial^2 b_{\alpha\beta}}{\partial u^2}. \tag{7}$$

It is easily seen that, if we now assume that σ^0 is reasonably bounded and smooth as $u \to \infty$, the 'Bondi news function', $\partial\sigma^0/\partial u$, on comparing (5) and (7), is given by

$$\frac{\partial\sigma^0}{\partial u} = \frac{1}{4}\eth n^{a}\eth n^{b}\frac{\partial b_{ab}}{\partial u}. \tag{8}$$

For slowly moving sources, it may be shown (see appendix) that

$$b^{ab} \simeq 2\left[\frac{\partial^2}{\partial\bar{x}^{0^2}}\int (T^{00}\bar{x}^{a}\bar{x}^{b})\,\mathrm{d}^3\bar{x} + n^{c}\frac{\partial^2}{\partial\bar{x}^{0^2}}\int (2T^{0(a}\bar{x}^{b)}\bar{x}^{c} - T^{0c}\bar{x}^{a}\bar{x}^{b})\,\mathrm{d}^3\bar{x}\right]_{|\bar{x}^0 = u} \tag{9}$$

On combining (9) with (8), we will then have

$$\frac{\partial\sigma^0}{\partial u} \simeq \frac{1}{2}\eth n^{a}\eth n^{b}\left[\frac{\partial^3}{\partial\bar{x}^{0^3}}\int (T^{00}\bar{x}^{a}\bar{x}^{b})\,\mathrm{d}^3\bar{x} + n^{c}\frac{\partial^3}{\partial\bar{x}^{0^3}}\int (2T^{0a}\bar{x}^{b}\bar{x}^{c} - T^{0c}\bar{x}^{a}\bar{x}^{b})\,\mathrm{d}^3\bar{x}\right]_{|\bar{x}^0 = u}. \tag{10}$$

4. Conclusion

We have investigated connections between harmonic and null coordinate systems, and via formula (10) we have obtained an approximate formula for the 'Bondi news function' in terms of physical quantities.

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Appendix

We define $\gamma^{\mu\nu}$ by $(-\bar{g})^{1/2}\bar{g}^{\mu\nu} = n^{\mu\nu}+\gamma^{\mu\nu}$, and assume there exists a small dimensionless parameter λ in which we may expand all quantities.

We assume that the sources of space–time are slowly moving, and as a consequence introduce, in the region of the sources, the inner variables $t = \lambda\bar{x}^0, \bar{x}^a$. Then in the source region we assume

$$\gamma^{00} = \lambda^2\, {}^{\text{in}}\underset{2}{\gamma}^{00} + \lambda^3\, {}^{\text{in}}\underset{3}{\gamma}^{00} + \ldots,$$

$$\gamma^{0a} = \lambda^3\, {}^{\text{in}}\underset{3}{\gamma}^{0a} + \ldots,$$

$$\gamma^{ab} = \lambda^4\, {}^{\text{in}}\underset{4}{\gamma}^{ab} + \lambda^5\, {}^{\text{in}}\underset{5}{\gamma}^{ab} + \ldots,$$

and for the energy momentum tensor $T^{\mu\nu}$

$$T^{00} = \lambda^2 \underset{2}{T}^{00} + \lambda^4 \underset{4}{T}^{00} + \ldots$$

$$T^{0a} = \lambda^3 \underset{3}{T}^{0a} + \lambda^5 \underset{5}{T}^{0a} + \ldots$$

$$T^{ab} = \lambda^4 \underset{4}{T}^{ab} + \lambda^6 \underset{6}{T}^{ab} + \ldots$$

where $\underset{n}{\gamma}^{\mu\nu}$ and $\underset{n}{T}^{\mu\nu}$ are functions of $(t, \bar{x}^a)$. The Einstein field equations

$$G^{\mu\nu} = -8\pi T^{\mu\nu} \tag{A.1}$$

using harmonic coordinates, yield

$$\nabla^2\, {}^{\text{in}}\underset{2}{\gamma}^{00} = -16\pi \underset{2}{T}^{00}, \tag{A.2}$$

$$\nabla^2\, {}^{\text{in}}\underset{3}{\gamma}^{00} = 0 \tag{A.3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\bar{x}^a\partial\bar{x}^a}.$$

We take as the solution to (A.2), (A.3)

$$\overset{\text{in}}{\underset{2}{\gamma}}{}^{00} = \int \frac{\underset{2}{T}{}^{00}_{|\bar{x}^a = \xi^a}\, \mathrm{d}^3\xi}{|\bar{x}^a - \xi^a|}, \tag{A.4}$$

$$\overset{\text{in}}{\underset{3}{\gamma}}{}^{00} = A^{00}, \tag{A.5}$$

where A^{00} is independent of $\bar{x}^a$.

On using (A.5) and harmonic coordinates, the field equations will further yield

$$\nabla^2 \underset{4}{\gamma}{}^{ab} = \frac{\delta^{ab}}{8} \frac{\partial \overset{\text{in}}{\underset{2}{\gamma}}{}^{00}}{\partial \bar{x}^d} \frac{\partial \overset{\text{in}}{\underset{2}{\gamma}}{}^{00}}{\partial \bar{x}^d} - \frac{1}{4} \frac{\partial \overset{\text{in}}{\underset{2}{\gamma}}{}^{00}}{\partial \bar{x}^a} \frac{\partial \overset{\text{in}}{\gamma}{}^{00}}{\partial \bar{x}^b} - 16\pi \underset{4}{T}{}^{ab}, \tag{A.6}$$

$$\nabla^2\, \overset{\text{in}}{\underset{5}{\gamma}}{}^{ab} = 0, \tag{A.7}$$

and integrability condition

$$\frac{\partial \underset{2}{T}{}^{00}}{\partial t} + \frac{\partial \underset{3}{T}{}^{0a}}{\partial \bar{x}^a} = 0. \tag{A.8}$$

For large $R = (\bar{x}^a \bar{x}^a)^{1/2}$, solutions to (A.6), (A.7) will take the form

$$\overset{\text{in}}{\underset{4}{\gamma}}{}^{ab} = \frac{2}{R} \frac{\partial^2}{\partial t^2}\left(\int \underset{2}{T}{}^{00} \bar{x}^a \bar{x}^b\, \mathrm{d}^3\bar{x} \right) + \frac{2n^c}{R^2} \frac{\partial}{\partial t}\left(\int (2\underset{3}{T}{}^{0a} \bar{x}^b \bar{x}^c - T^{0c} \bar{x}^a \bar{x}^b)\, \mathrm{d}^3\bar{x} \right) + \frac{M^2 n^a n^b}{R^2} + \mathrm{O}(R^{-3}), \tag{A.9}$$

$$\underset{5}{\gamma}{}^{ab} = A^{ab}, \tag{A.10}$$

where $M = \int \underset{2}{T}{}^{00}\, \mathrm{d}^3\bar{x} =$ constant (by A.8), and A^{ab} is independent of $\bar{x}^a$.

Far from the sources, we must treat derivatives with respect to $\bar{x}^0$ and $\bar{x}^a$ on the same basis. Consequently, in this far region, we introduce the far variables $t = \lambda x^0$, $X^a = \lambda \bar{x}^a$. We assume that in this outer region $\gamma^{\mu\nu}$ may be expanded in λ as follows

$$\gamma^{00} = \lambda^3\, \overset{\text{far}}{\underset{3}{\gamma}}{}^{00} + \lambda^4\, \overset{\text{far}}{\underset{4}{\gamma}}{}^{00} + \dots$$

$$\gamma^{0a} = \lambda^4\, \overset{\text{far}}{\underset{4}{\gamma}}{}^{0a} + \dots$$

$$\gamma^{ab} = \lambda^5\, \overset{\text{far}}{\underset{5}{\gamma}}{}^{ab} + \lambda^6\, \overset{\text{far}}{\underset{6}{\gamma}}{}^{ab} + \dots,$$

where ${}^{\text{far}}\gamma^{\mu\nu}$ are functions of (t, X^a).

In the outer region the field equations

$$G^{\mu\nu} = 0,$$

on using harmonic coordinates, yield

$$\Box^2 \, {}^{\mathrm{far}}\underset{3}{\gamma}^{00} = 0, \tag{A.11}$$

$$\Box^2 \, {}^{\mathrm{far}}\underset{5}{\gamma}^{ab} = 0, \tag{A.12}$$

$$\Box^2 \, {}^{\mathrm{far}}\underset{6}{\gamma}^{ab} = \frac{1}{4} \frac{\partial^{\mathrm{far}}\underset{3}{\gamma}^{00}}{\partial X^a} \frac{\partial^{\mathrm{far}}\underset{3}{\gamma}^{00}}{\partial X^a} - \frac{\delta^{ab}}{8}\left[3\left(\frac{\partial^{\mathrm{far}}\underset{3}{\gamma}^{00}}{\partial t}\right) + \frac{\partial^{\mathrm{far}}\underset{3}{\gamma}^{00}}{\partial X^a} \frac{\partial^{\mathrm{far}}\underset{3}{\gamma}^{00}}{\partial X^a}\right], \tag{A.13}$$

where

$$\Box^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^a \partial X^a}.$$

We assume that ${}^{\mathrm{far}}\underset{3}{\gamma}^{00}$ is sufficiently described by a monopole, then

$$\underset{3}{\gamma}^{00} = \frac{W^{00}}{D}, \tag{A.14}$$

where $D = (X^a X^a)^{1/2}$, and W^{00} depends on (t, X^a) through $(t - D)$. On expanding in ${}^{\mathrm{in}}\underset{2}{\gamma}^{00}$ for large R

$${}^{\mathrm{in}}\underset{2}{\gamma}^{00} = \frac{M}{R} + \mathrm{O}(R^{-2}),$$

and so on matching (Burke 1971) the solutions obtained so far for the inner and far fields, we obtain that $W^{00} = M$.

We take solutions to (A.12), (A.13) to be

$${}^{\mathrm{far}}\underset{5}{\gamma}^{ab} = \frac{W_1^{ab}}{D} + n^c\left(\frac{V_1^{abc\prime}}{D} + \frac{V_1^{abc}}{D^2}\right), \tag{A.15}$$

$${}^{\mathrm{far}}\underset{6}{\gamma}^{ab} = \frac{W_2^{ab}}{D} + n^c\left(\frac{V_2^{abc\prime}}{D} + \frac{V_2^{abc}}{D^2}\right) + \frac{M^2 n^a n^b}{D^2} \tag{A.16}$$

where $W_1^{ab}, W_2^{ab}, V_1^{abc}, V_2^{abc}$ depend on (t, X^a) through $(t - D)$, and the prime denotes differentiation with respect to $(t - D)$. It should be noted that for correct matching with the far far field solutions (4) one must neglect advanced solutions in obtaining (A.15) and (A.16). Matching up (Burke 1971) expressions for the inner and far fields obtained so far, we find that

$$W_1^{ab} = 2\left(\frac{\partial^2}{\partial t^2} \int \underset{2}{T}^{00} \, \mathrm{d}^3\bar{x}\right)_{|t = t - D},$$

$$V_1^{abc} = 0, \qquad W_2^{ab} = 0, \tag{A.17}$$

and

$$V_2^{abc} = 2\left[\frac{\partial}{\partial t}\int (2\underset{3}{T}^{0(a}\bar{x}^{b)} - \underset{3}{T}^{0c}\bar{x}^a\bar{x}^b) \, \mathrm{d}^3\bar{x}\right]_{|t = t - D}$$

We limit the behaviour of ΔQ so that $\int_\infty^u \Delta Q = \mathrm{O}(\lambda^7)$, and then the trivial matching of the far region and the far far region solutions yields the approximation to b^{ab} given by formula (9).

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